Hierarchical triple systems in Newtonian and post-Newtonian gravity

Clifford Will

University of Florida & Institut d'Astrophysique de Paris

GReCO Seminar, IAP, 12 September 2022

Adventures of a general relativist in an N-body world

- Brief history of the 3-body problem
- Hierarchical triple systems
- Higher multipoles and quadrupole-squared terms
- A companion black hole for SgrA*?
- Quadrupole-PN cross-terms and Mercury's perihelion
- Ongoing and future work

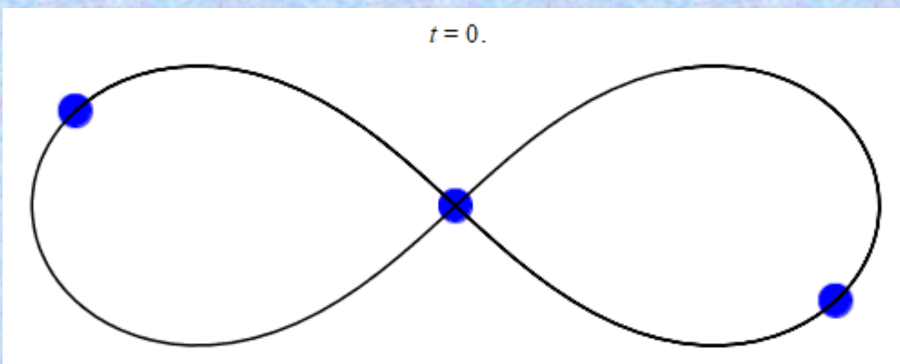
Brief history of the 3-body problem

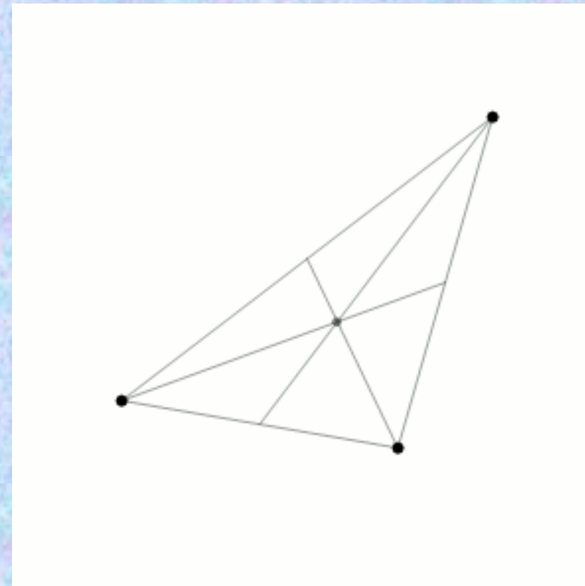
- Lunar apsides
  - 1687 – Newton and the lunar apsides
  - 1750 – Prize-winning solution by Clairaut
  - 1878 – Hill
- The problem of Mercury
  - 1846 - Leverrier (and Adams) and Neptune
  - 1859 - Mercury's discrepancy and Vulcan
  - 1912–15 - Einstein's solution
- The general 3-body problem
  - 1887 - King Oscar II of Sweden and the 3-body prize
  - 1890–95 - Henri Poincaré's flawed attempt
  - 1913 – Sundman's solution in powers of $t^{1/3}$

FLORIDA GRAVITY

Brief history of the 3-body problem

- Special solutions

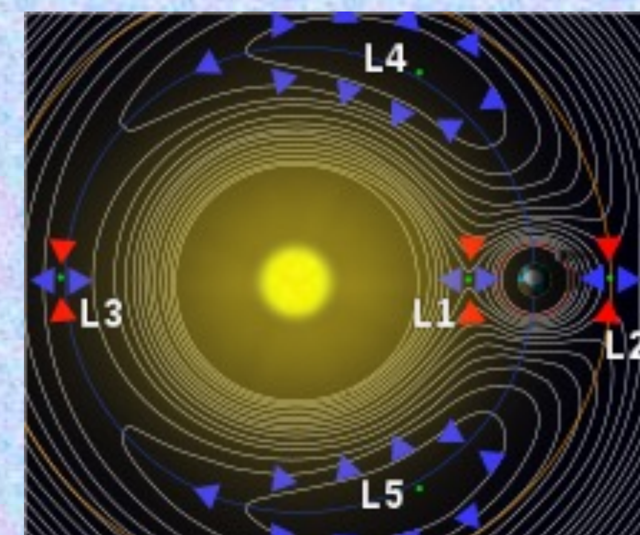




- Restricted 3-body problem (m_3 = 0)

Sun-Jupiter: Trojan asteroids at L_4 & L_5

Sun-Earth:
L_1 – SOHO, LISA Pathfinder, Athena
L_2 – WMAP, Planck, GAIA, Webb
L_4 & L_5 – dust clouds, 2 asteroids



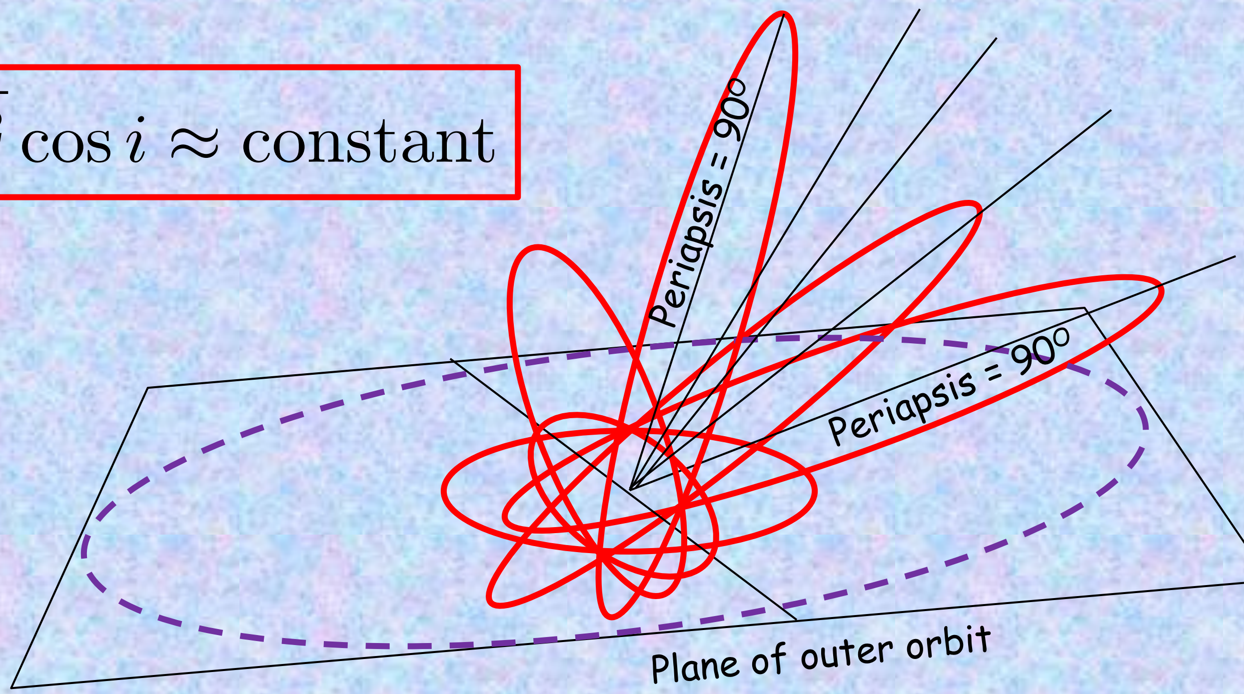
Brief history of the 3-body problem

The Kozai-Lidov mechanism - 1961

Lidov – spacecraft and moons

Kozai – asteroids and comets

$$\sqrt{1 - e^2} \cos i \approx \text{constant}$$



Hierarchical triple systems

- Two-body inner orbit, third body at large distance (A >> a)
- Expand perturbing potential in powers of a/A
- Quadrupole order $(a/A)^3$ – Kozai-Lidov oscillations
  - Interchange between inclination and eccentricity
  - KL resonances

$$\cos\iota\sqrt{1-e^2} \propto L_z = \text{constant}$$

- Octupole order $(a/A)^4$ – extreme eccentricity & flips
  - Krymolowski & Mazeh 1999
  - Ford, Kozinsky & Rasio 2000
  - Blaes, Lee & Socrates 2002
- Hot Jupiters
  - Naoz et al 2011
- Kozai mechanism and BBH formation?

Hierarchical triple systems

$$a^j = -\frac{Gmn^j}{r^2} + \frac{Gm_3}{R^2}\sum_{\ell=1}^{\infty}\frac{(2\ell+1)!!}{\ell!}\left(\frac{r}{R}\right)^{\ell} \times n^L N^{\langle jL\rangle}\left[\alpha_2^{\ell} - (-\alpha_1)^{\ell}\right] ,$$

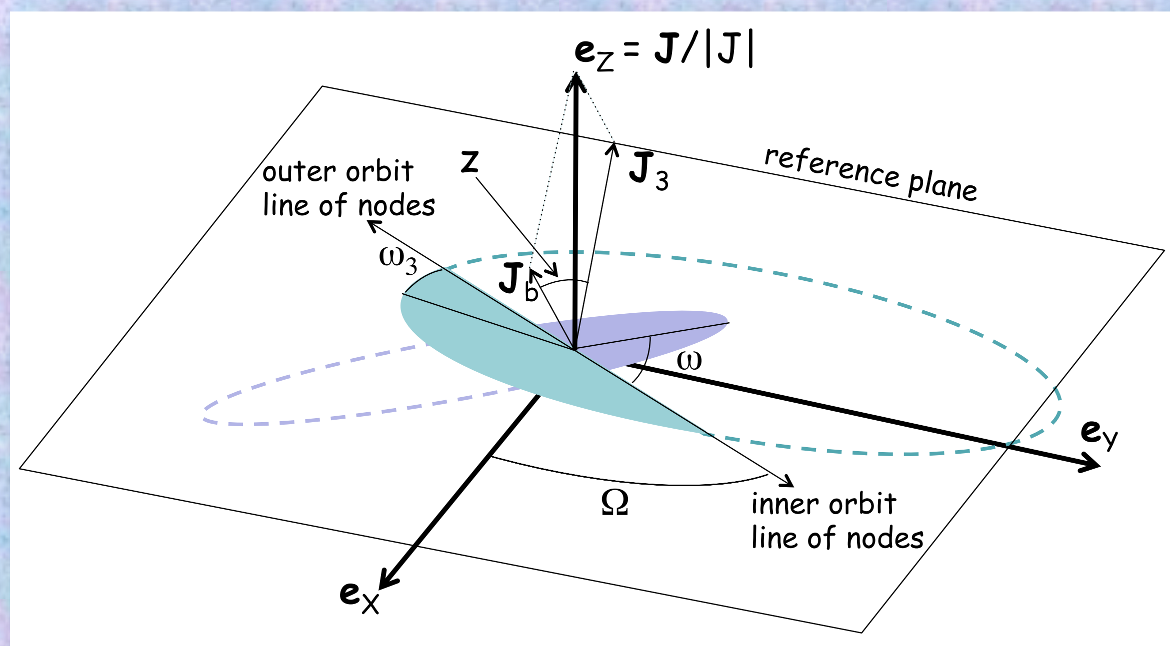
$$A^j = -\frac{GMN^j}{R^2} - \eta\frac{GMr}{R^3}\sum_{\ell=1}^{\infty}\frac{(2\ell+3)!!}{(\ell+1)!}\left(\frac{r}{R}\right)^{\ell} \times n^{L+1} N^{\langle j(L+1)\rangle}\left[\alpha_2^{\ell} - (-\alpha_1)^{\ell}\right] ,$$

$\ell = 1$ quadrupole

$\ell = 2$ octupole

$\ell = 3$ hexadecapole

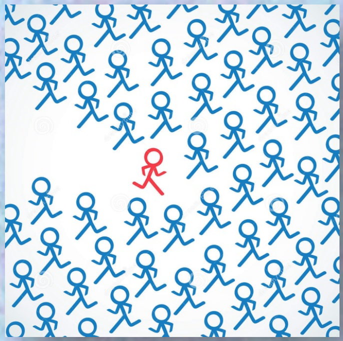
$\ell = 4$ dotriocontupole



Osculating orbit elements

"Lagrange planetary equations"

$$r = \frac{p}{1 + e\cos f}$$



$$\frac{dp}{dt} = 2\sqrt{\frac{p^3}{Gm}}\frac{1}{1+e\cos f}\mathcal{S},$$

$$\frac{de}{dt} = \sqrt{\frac{p}{Gm}}\left[\sin f\,\mathcal{R} + \frac{2\cos f + e(1+\cos^2 f)}{1+e\cos f}\mathcal{S}\right],$$

$$\frac{d\iota}{dt} = \sqrt{\frac{p}{Gm}}\frac{\cos(\omega+f)}{1+e\cos f}\mathcal{W},$$

$$\sin\iota\frac{d\Omega}{dt} = \sqrt{\frac{p}{Gm}}\frac{\sin(\omega+f)}{1+e\cos f}\mathcal{W},$$

$$\frac{d\omega}{dt} = \frac{1}{e}\sqrt{\frac{p}{Gm}}\left[-\cos f\,\mathcal{R} + \frac{2+e\cos f}{1+e\cos f}\sin f\,\mathcal{S} - e\cot\iota\frac{\sin(\omega+f)}{1+e\cos f}\mathcal{W}\right]$$

$$\frac{df}{dt} = \frac{\sqrt{mp}}{r^2} - \left(\frac{d\omega}{dt} + \cos i\frac{d\Omega}{dt}\right)$$

Secular evolution of orbit elements

Quadrupole order (da/dt = dA/dt = 0 at all orders)

$$\frac{de}{d\tau} = \frac{15\pi}{2}\alpha\epsilon^3 \frac{e(1-e^2)^{1/2}}{(1-E^2)^{3/2}} \sin^2 z \sin\omega\cos\omega\,,$$

$$\frac{d\iota}{d\tau} = -\frac{15\pi}{2}\alpha\epsilon^3 \frac{e^2}{(1-e^2)^{1/2}(1-E^2)^{3/2}} \sin z \cos z \sin\omega\cos\omega\,,$$

$$\frac{d\Omega}{d\tau} = -\frac{3\pi}{2}\alpha\epsilon^3 \frac{1}{(1-e^2)^{1/2}(1-E^2)^{3/2}} \frac{\sin z \cos z}{\sin\iota} \left(1+4e^2-5e^2\cos^2\omega\right)\,,$$

$$\frac{d\varpi}{d\tau} = \frac{3\pi}{2}\alpha\epsilon^3 \frac{(1-e^2)^{1/2}}{(1-E^2)^{3/2}} \left[1-\sin^2 z\left(4-5\cos^2\omega\right)\right]\,,$$

$$\frac{dE}{d\tau} = 0\,,$$

$$\frac{d\iota_3}{d\tau} = -\frac{15\pi}{2}\eta(1+\alpha)^{1/2}\epsilon^{7/2} \frac{e^2}{(1-E^2)^2} \sin z \sin\omega\cos\omega\,,$$

$$\frac{d\varpi_3}{d\tau} = \frac{3\pi}{4}\eta(1+\alpha)^{1/2}\epsilon^{7/2} \frac{1}{(1-E^2)^2} \left[2+3e^2-3\sin^2 z\left(1+4e^2-5e^2\cos^2\omega\right)\right]\,.$$

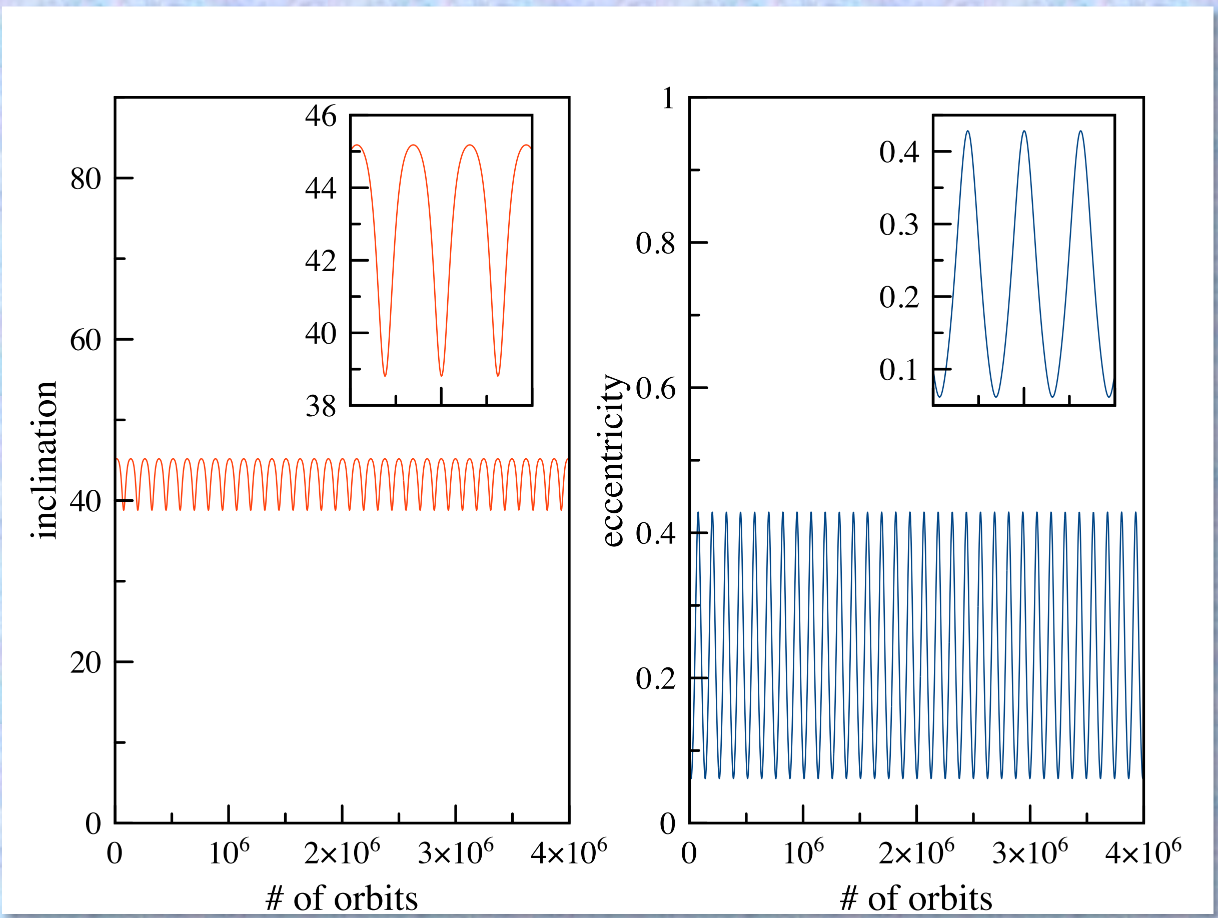
$$\tau = t/P_{\rm binary}$$

$$\alpha \equiv \frac{m_3}{m}\,, \quad \epsilon \equiv \frac{a}{A}\,, \quad \eta \equiv \frac{m_1 m_2}{m^2}\,, \quad \Delta \equiv \frac{m_2 - m_1}{m}\,.$$

Standard Kozai-Lidov oscillations when $\eta = 0$

Secular evolution of orbit elements

Quadrupole order (da/dt = dA/dt = 0 at all orders)



$\eta = 10^{-3}$; $m_3 = m/26$; $a = 6$; $A = 100$; $i = 45^o$; $e = 0.1$

Secular evolution of orbit elements

Octupole order

$$\frac{de}{d\tau} = -\frac{15\pi}{256}\alpha\epsilon^4\Delta\frac{E(1-e^2)^{1/2}}{(1-E^2)^{5/2}}\Big((4+3e^2)\left[(1+\cos z)(1+10\cos z-15\cos^2 z)\sin(\omega-\omega_3)\right.$$
$$\left.+(1-\cos z)(1-10\cos z-15\cos^2 z)\sin(\omega+\omega_3)\right]$$
$$-105e^2\sin^2 z\left[(1+\cos z)\sin(3\omega-\omega_3)+(1-\cos z)\sin(3\omega+\omega_3)\right]\Big)$$

$$\frac{de}{d\tau} = -\frac{15\pi}{256}\alpha\epsilon^4\Delta\frac{E(1-e^2)^{1/2}}{(1-E^2)^{5/2}}\cdots$$

$$-(11+10\cos z-45\cos^2 z)\cos(\omega+\omega_3)]$$
$$-35e^2\left[(1+\cos z)(1-3\cos z)\cos(3\omega-\omega_3)-(1-\cos z)(1+3\cos z)\cos(3\omega+\omega_3)\right]\Big),$$

$$\frac{d\varpi}{d\tau} = -\frac{15\pi}{256}\alpha\epsilon^4\Delta\frac{E(1-e^2)^{1/2}}{e(1-E^2)^{5/2}}\Big((4+9e^2)\left[(1+\cos z)(1+10\cos z-15\cos^2 z)\cos(\omega-\omega_3)\right.$$
$$\left.+(1-\cos z)(1-10\cos z-15\cos^2 z)\cos(\omega+\omega_3)\right]$$
$$-105e^2\sin^2 z\left[(1+\cos z)\cos(3\omega-\omega_3)+(1-\cos z)\cos(3\omega+\omega_3)\right]\Big),$$

$$\frac{dE}{d\tau} = \frac{15\pi}{256}\eta(1+\alpha)^{1/2}\epsilon^{9/2}\Delta\frac{e}{(1-E^2)^2}\Big((4+3e^2)\left[(1+\cos z)(1+10\cos z-15\cos^2 z)\sin(\omega-\omega_3)\right.$$
$$\left.-(1-\cos z)(1-10\cos z-15\cos^2 z)\sin(\omega+\omega_3)\right]$$
$$-35e^2\sin^2 z\left[(1+\cos z)\sin(3\omega-\omega_3)-(1-\cos z)\sin(3\omega+\omega_3)\right]\Big),$$

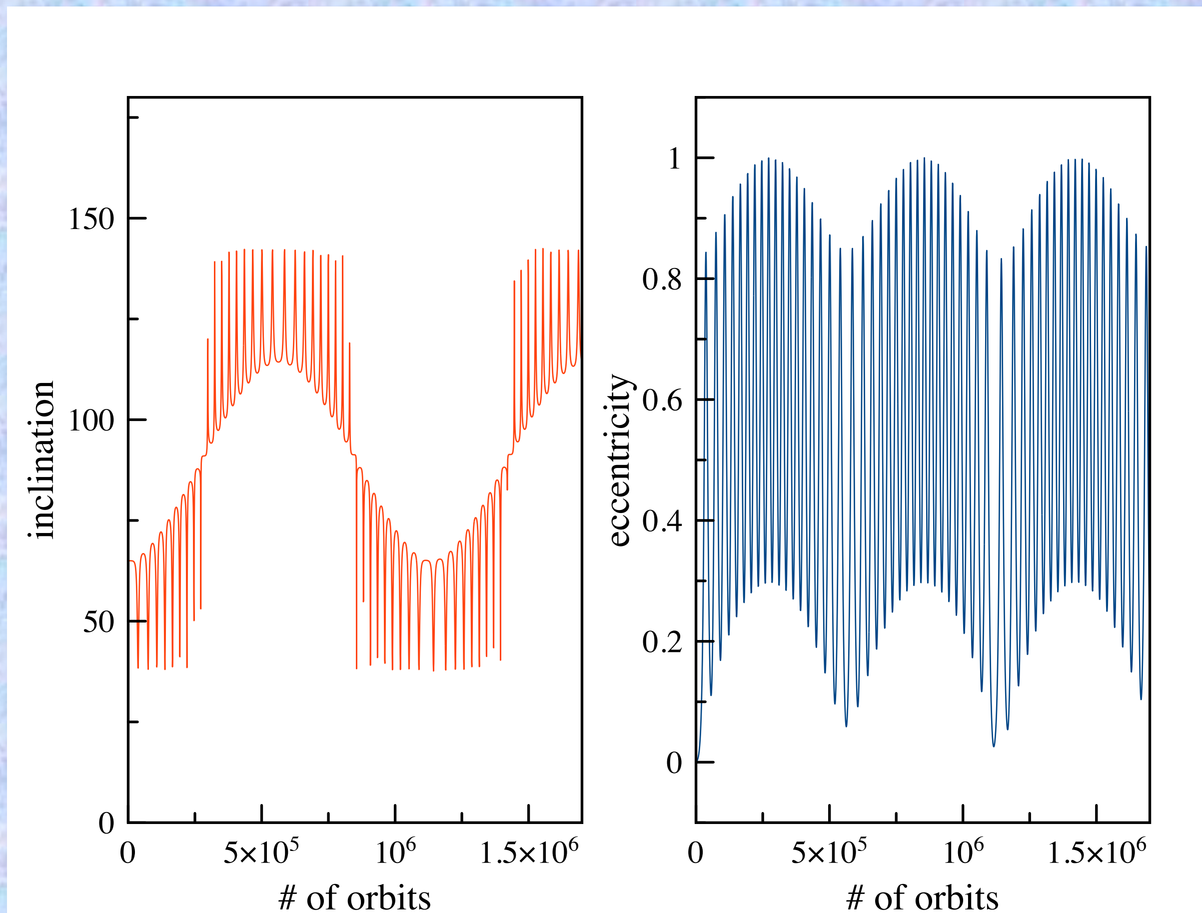
$$\frac{d\iota_3}{d\tau} = \frac{15\pi}{256}\eta(1+\alpha)^{1/2}\epsilon^{9/2}\Delta\frac{Ee}{(1-E^2)^3}\sin z\Big((4+3e^2)\left[(1+10\cos z-15\cos^2 z)\sin(\omega-\omega_3)\right.$$
$$\left.+(1-10\cos z-15\cos^2 z)\sin(\omega+\omega_3)\right]$$
$$-35e^2\left[(1+\cos z)(3-\cos z)\sin(3\omega-\omega_3)+(1-\cos z)(3+\cos z)\sin(3\omega+\omega_3)\right]\Big),$$

$$\frac{d\varpi_3}{d\tau} = -\frac{15\pi}{256}\eta(1+\alpha)^{1/2}\epsilon^{9/2}\Delta\frac{e(1+4E^2)}{E(1-E^2)^3}\Big((4+3e^2)\left[(1+\cos z)(1+10\cos z-15\cos^2 z)\cos(\omega-\omega_3)\right.$$
$$\left.+(1-\cos z)(1-10\cos z-15\cos^2 z)\cos(\omega+\omega_3)\right]$$
$$-35e^2\sin^2 z\left[(1+\cos z)\cos(3\omega-\omega_3)+(1-\cos z)\cos(3\omega+\omega_3)\right]\Big).$$

Agrees with
Naoz et al

Secular evolution of orbit elements

Octupole order



$\eta=10^{-3}$; $m_3 = m/26$; $a = 6$; $A = 100$; $i = 65^o$; $e = 0.001$; $E = 0.6$

FLORIDA GRAVITY

Secular evolution of orbit elements

Hexadecapole order

$$\frac{de}{d\tau} = -\frac{315\pi}{1024}\alpha\epsilon^5(1-3\eta)\frac{e(1-e^2)^{1/2}}{(1-E^2)^{7/2}}$$
$$\times\Big((2+3E^2)\sin^2 z\left[(4+2e^2)(1-7\cos^2 z)\sin 2\omega - 21e^2\sin^2 z\sin 4\omega\right]$$
$$- E^2\Big\{(4+2e^2)\left[(1+\cos z)^2(1-7\cos z+7\cos^2 z)\sin(2\omega-2\omega_3)\right.$$
$$+ 21e^2\left[(1-2\cos z)(1+\cos z)^2\sin(4\omega-2\omega_3) - (1+2\cos z)(1-\cos z)^2\sin(4\omega+2\omega_3)\right]\Big\}\Big),$$

$$\frac{de}{d\tau} = -\frac{315\pi}{1024}\alpha\epsilon^5(1-3\eta)\ldots$$

$$\frac{d\Omega}{d\tau} = \frac{45\pi}{2048}\alpha\epsilon^5(1-3\eta)\frac{1}{(1-e^2)^{1/2}(1-E^2)^{7/2}}\frac{\sin z}{\sin\iota}$$
$$\times\Big(2(2+3E^2)\cos z\left[(8+40e^2+15e^4)(3-7\cos^2 z) - 28e^2(2+e^2)(4-7\cos^2 z)\cos 2\omega + 147e^4\sin^2 z\cos\right.$$
$$- 4E^2\cos z(4-7\cos^2 z)(8+40e^2+15e^4)\cos 2\omega_3$$
$$+ 7E^2e^2\Big\{2(2+e^2)\left[(1+\cos z)(5+7\cos z-28\cos^2 z)\cos(2\omega-2\omega_3)\right.$$
$$\left.-(1-\cos z)(5-7\cos z-28\cos^2 z)\cos(2\omega+2\omega_3)\right]$$
$$- 21e^2\left[(1-2\cos z)(1+\cos z)^2\cos(4\omega-2\omega_3) - (1+2\cos z)(1-\cos z)^2\cos(4\omega+2\omega_3)\right]\Big\}\Big),$$

$$\frac{d\varpi}{d\tau} = \frac{45\pi}{1024}\alpha\epsilon^5(1-3\eta)\frac{(1-e^2)^{1/2}}{(1-E^2)^{7/2}}$$
$$\times\Big((2+3E^2)\left[(4+3e^2)(3-30\cos^2 z+35\cos^4 z) - 28(1+e^2)\sin^2 z(1-7\cos^2 z)\cos 2\omega + 147e^2\sin^4 z\right.$$
$$- 10E^2(4+3e^2)\sin^2 z(1-7\cos^2 z)\cos 2\omega_3$$
$$\ldots + 7\cos^2 z)\cos(2\omega-2\omega_3)$$
$$\ldots (1-\cos z)^2\cos(4\omega+2\omega_3)\right]\Big\}\Big),$$

$$\ldots 7\cos^2 z)\sin(2\omega-2\omega_3)$$
$$\left.-(1-\cos z)^2(1+7\cos z+7\cos^2 z)\sin(2\omega+2\omega_3)\right]$$
$$+ 147e^4\sin^2 z\left[(1+\cos z)^2\sin(4\omega-2\omega_3) - (1-\cos z)^2\sin(4\omega+2\omega_3)\right]\Big),$$

$$\frac{d\iota_3}{d\tau} = \frac{45\pi}{2048}\eta(1-3\eta)(1+\alpha)^{1/2}\epsilon^{11/2}\frac{\sin z}{(1-E^2)^4}$$
$$\times\Big(14e^2(2+3E^2)\left[(4+2e^2)(1-7\cos^2 z)\sin 2\omega - 21e^2\sin^2 z\sin 4\omega\right]$$
$$+ 2E^2(8+40e^2+15e^4)\cos z(1-7\cos^2 z)\sin 2\omega_3$$
$$- 7E^2e^2\Big\{4(2+e^2)\left[(1+\cos z)(1-7\cos z+7\cos^2 z)\sin(2\omega-2\omega_3)\right.$$
$$\left.+(1-\cos z)(1+7\cos z+7\cos^2 z)\sin(2\omega+2\omega_3)\right]$$
$$+ 21e^2\left[(2-\cos z)(1+\cos z)^2\sin(4\omega-2\omega_3) + (2+\cos z)(1-\cos z)^2\sin(4\omega+2\omega_3)\right]\Big\}\Big),$$

$$\frac{d\varpi_3}{d\tau} = \frac{45\pi}{4096}\eta(1-3\eta)(1+\alpha)^{1/2}\epsilon^{11/2}\frac{1}{(1-E^2)^4}$$
$$\times\Big((4+3E^2)\left[(8+40e^2+15e^4)(3-30\cos^2 z+35\cos^4 z)\right.$$
$$\left.-140e^2(2+e^2)\sin^2 z(1-7\cos^2 z)\cos 2\omega + 735e^4\sin^4 z\cos 4\omega\right]$$
$$- (2+5E^2)\Big\{2(8+40e^2+15e^4)\sin^2 z(1-7\cos^2 z)\cos 2\omega_3$$
$$- 28e^2(2+e^2)\left[(1+\cos z)^2(1-7\cos z+7\cos^2 z)\cos(2\omega-2\omega_3)\right.$$
$$\left.+(1-\cos z)^2(1+7\cos z+7\cos^2 z)\cos(2\omega+2\omega_3)\right]$$
$$- 147e^4\sin^2 z\left[(1+\cos z)^2\cos(4\omega-2\omega_3) + (1-\cos z)^2\cos(4\omega+2\omega_3)\right]\Big\}\Big).$$

General relativity

Add the standard pericenter advances

$$\frac{d\varpi}{dt} = 6\pi \frac{Gm}{c^2 a(1-e^2)P_{\rm inner}}\,,$$

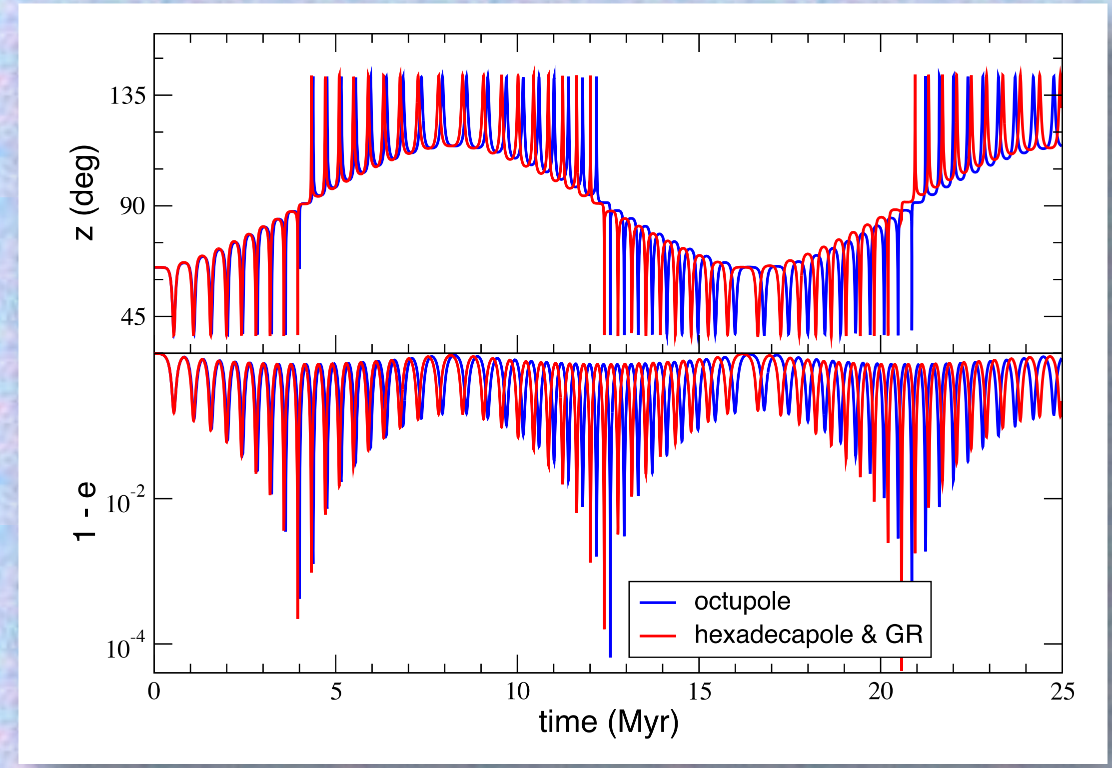
$$\frac{d\varpi_3}{dt} = 6\pi \frac{GM}{c^2 A(1-E^2)P_{\rm outer}}\,.$$

New parameter

$$\delta = \frac{Gm}{c^2 a}$$

Case studies

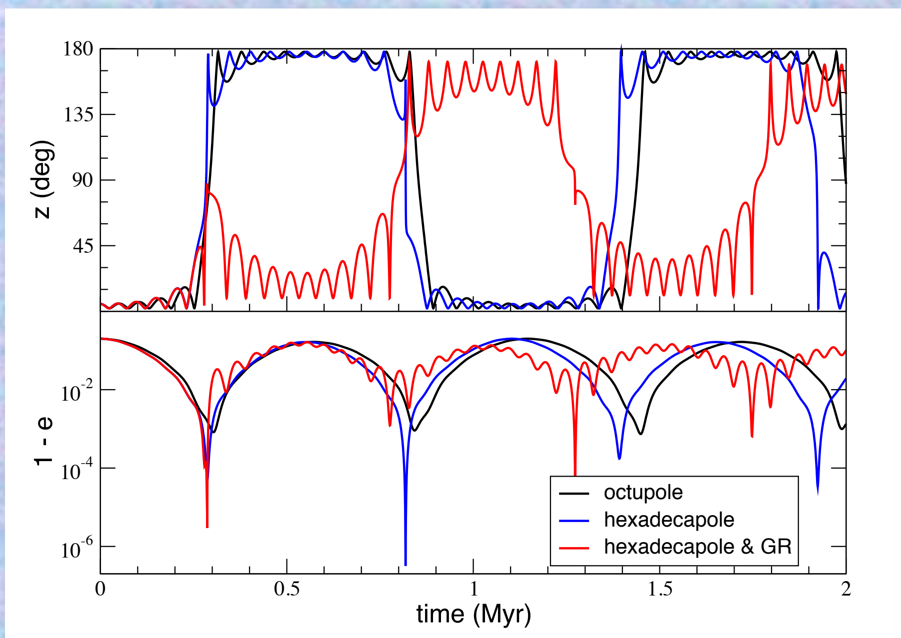
System	m_1	m_2	m_3	a (a.u.)	A (a.u.)	e	E	z	ω	ω_3
Hot Jupiters	M_J	$M_\odot$	$40\,M_J$	6	100	0.001	0.6	65	45	0
Coplanar Flips	M_J	$M_\odot$	$0.03\,M_\odot$	4	50	0.8	0.6	5	0	0
Equal Masses	$1.4M_\odot$	$1.4M_\odot$	$50M_\odot$	7	80	0.99	0.6	5	45	0

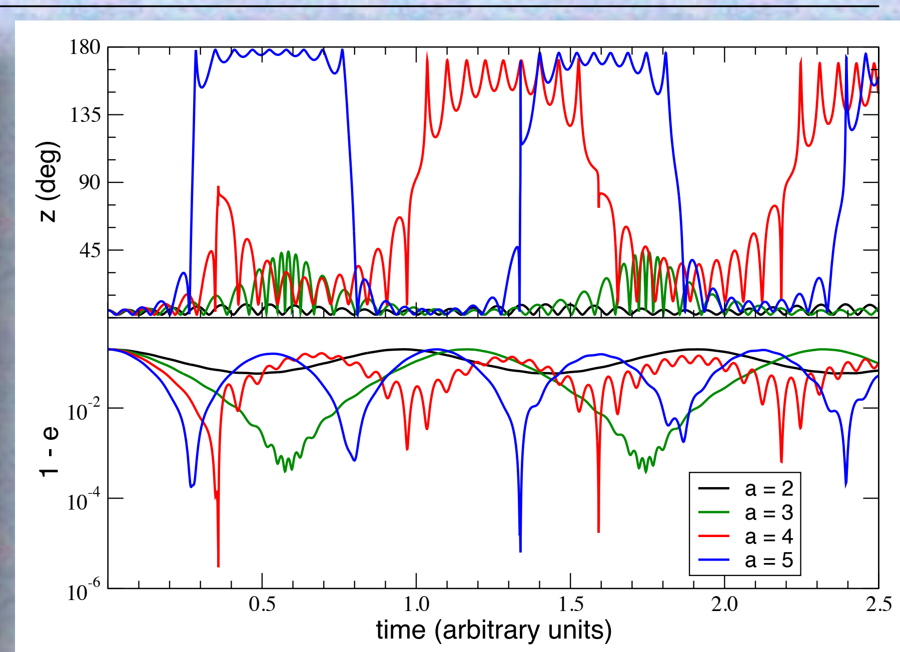


CMW, PRD 96, 023017 (2017)

Case studies

System	m_1	m_2	m_3	a (a.u.)	A (a.u.)	e	E	z	ω	ω_3
Hot Jupiters	M_J	$M_\odot$	40 M_J	6	100	0.001	0.6	65	45	0
Coplanar Flips	M_J	$M_\odot$	0.03 $M_\odot$	4	50	0.8	0.6	5	0	0
Equal Masses	$1.4M_\odot$	$1.4M_\odot$	$50M_\odot$	7	80	0.99	0.6	5	45	0



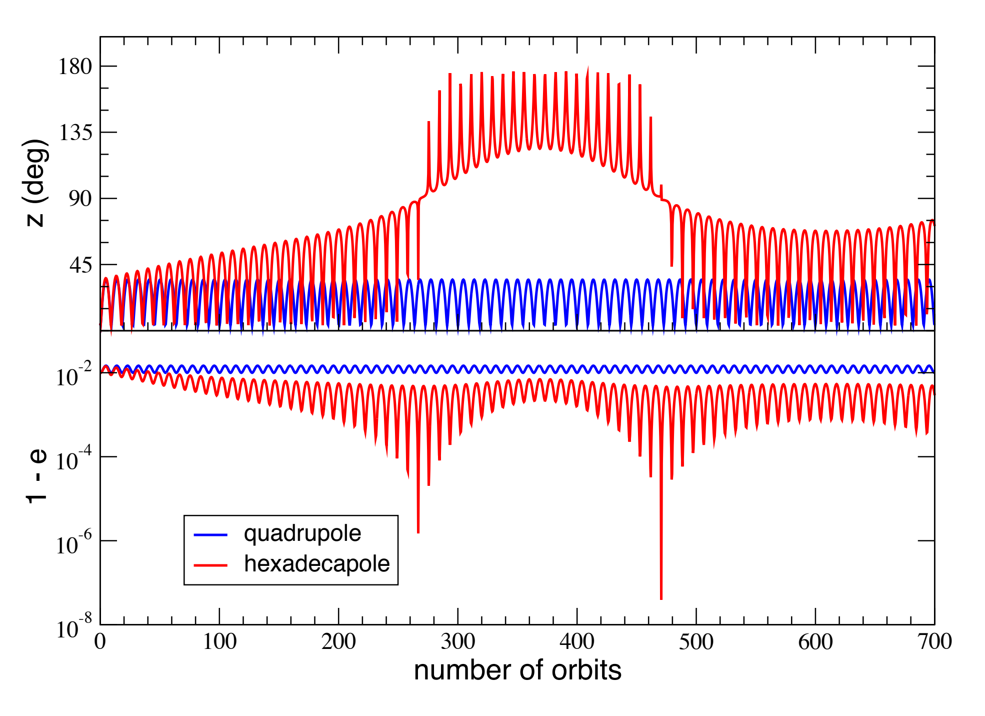


Li et al 2014

Note: flips suppressed when initial pericenters point in the same direction

Case studies

System	m_1	m_2	m_3	a (a.u.)	A (a.u.)	e	E	z	ω	ω_3
Hot Jupiters	M_J	$M_\odot$	$40\,M_J$	6	100	0.001	0.6	65	45	0
Coplanar Flips	M_J	$M_\odot$	$0.03\,M_\odot$	4	50	0.8	0.6	5	0	0
Equal Masses	$1.4M_\odot$	$1.4M_\odot$	$50M_\odot$	7	80	0.99	0.6	5	45	0



Hierarchies and 2nd order terms

1st order, secular approx

$$\frac{m_3}{m}\left(\frac{a}{A}\right)^3$$ quadrupole

$$\frac{m_3}{m}\left(\frac{a}{A}\right)^4$$ octupole

$$\frac{m_3}{m}\left(\frac{a}{A}\right)^5$$ hexadecapole

$$\frac{m_3}{m}\left(\frac{a}{A}\right)^6$$ dotriocontopole

2nd order, beyond secular

$$\frac{(m_3/m)^2}{\sqrt{1+m_3/m}}\left(\frac{a}{A}\right)^{9/2}$$ Q^2 terms

$$\left(\frac{m_3}{m}\right)^2\left(\frac{a}{A}\right)^6$$ Q^2 terms

Two-timescale analysis

$$\frac{dX_\alpha(t)}{dt} = \varepsilon Q_\alpha(X_\beta(t), t)$$

$$\theta = \varepsilon t \,, \qquad \frac{d}{dt} = \varepsilon \frac{\partial}{\partial \theta} + \frac{\partial}{\partial t}$$

$$X_\alpha(\theta, t) \equiv \tilde{X}_\alpha(\theta) + \varepsilon Y_\alpha(\tilde{X}_\beta(\theta), t),$$

$$\frac{d\tilde{X}_\alpha}{d\theta} = \langle Q_\alpha(\tilde{X}_\beta + \varepsilon Y_\beta, t)\rangle \,,$$

$$\frac{\partial Y_\alpha}{\partial t} = \mathcal{AF}\left(Q_\alpha(\tilde{X}_\beta + \varepsilon Y_\beta, t)\right) - \varepsilon \frac{\partial Y_\alpha}{\partial \tilde{X}_\gamma} \frac{d\tilde{X}_\gamma}{d\theta}$$

$$\frac{d\tilde{X}_\alpha}{dt} = \varepsilon \left\langle Q_\alpha^{(0)} \right\rangle + \varepsilon^2 \left\langle \mathcal{AF}\left(\frac{\partial Q_\alpha^{(0)}}{\partial \tilde{X}_\beta}\right) \int_0^t \mathcal{AF}\left(Q_\beta^{(0)}\right) dt' \right\rangle$$

FLORIDA GRAVITY

The secular approximation

A, B, … vary on short orbital timescale P_{in}
M,N, … vary on long orbital timescale P_{out}

$$\langle A\, M \rangle = \langle A \rangle \langle M \rangle + O(P_{\rm in}/P_{\rm out})^2$$

P_{out} x (AMBN)

$$\left\langle \mathcal{AF}(AM) \int_0^t \mathcal{AF}(BN) dt' \right\rangle = \langle A \rangle \langle B \rangle \left\langle \mathcal{AF}(M) \int_0^t \mathcal{AF}(N) dt' \right\rangle + \left\langle \mathcal{AF}(A) \int_0^t \mathcal{AF}(B) dt' \right\rangle \langle MN \rangle + O[P_{\rm in}^2/P_{\rm out} \times \langle AMBN \rangle]$$

P_{in} x (AMBN)

Hierarchies and 2nd order terms

1st order, secular approx

$$\frac{m_3}{m}\left(\frac{a}{A}\right)^3$$ quadrupole

$$\frac{m_3}{m}\left(\frac{a}{A}\right)^4$$ octupole

$$\frac{m_3}{m}\left(\frac{a}{A}\right)^5$$ hexadecapole

$$\frac{m_3}{m}\left(\frac{a}{A}\right)^6$$ dotriocontopole

2nd order, beyond secular

$$\frac{(m_3/m)^2}{\sqrt{1+m_3/m}}\left(\frac{a}{A}\right)^{9/2}$$ Q^2 terms

$$\left(\frac{m_3}{m}\right)^2\left(\frac{a}{A}\right)^6$$ Q^2 terms

Evolution of the orbit elements

CMW, PRD 103, 063003 (2021)

$$\frac{de}{d\tau} = \frac{15\pi}{32}\frac{\alpha^2\epsilon^{9/2}}{(1+\alpha)^{1/2}}$$

$$\frac{de}{d\tau} = \frac{15\pi}{\cdots}\;\frac{\alpha^2\epsilon^{9/2}}{\cdots}\;\frac{e(1-e^2)}{\cdots}\Big[\cdots \quad \cdots\cos z)\sin(2\omega+2\omega_3))\Big],$$

$$\frac{d\iota}{d\tau} = \cdots \quad \cdots 2+3\cos z)\sin(2\omega+2\omega_3)$$

$$-\,2(2-17e^2)\cos(z)\sin(2\omega_3)\Big)\Big],$$

$$\frac{d\Omega}{d\tau} = -\frac{3\pi}{64}\frac{\alpha^2\epsilon^{9/2}}{(1+\alpha)^{1/2}}\frac{1}{(1-E^2)^3}\frac{\sin(z)}{\sin(i)}\Big[(3+2E^2)\Big(2+33e^2-3(2-17e^2)\cos^2 z+15e^2(1-3\cos^2 z)\cos 2\omega\Big)$$
$$-\frac{5}{2}E^2H(E)\Big(5e^2(1+\cos z)(1-9\cos z)\cos(2\omega-2\omega_3)+5e^2(1-\cos z)(1+9\cos z)\cos(2\omega+2\omega_3)$$
$$+\,2(2-17e^2)(1-3\cos^2 z)\cos 2\omega_3\Big)\Big],$$

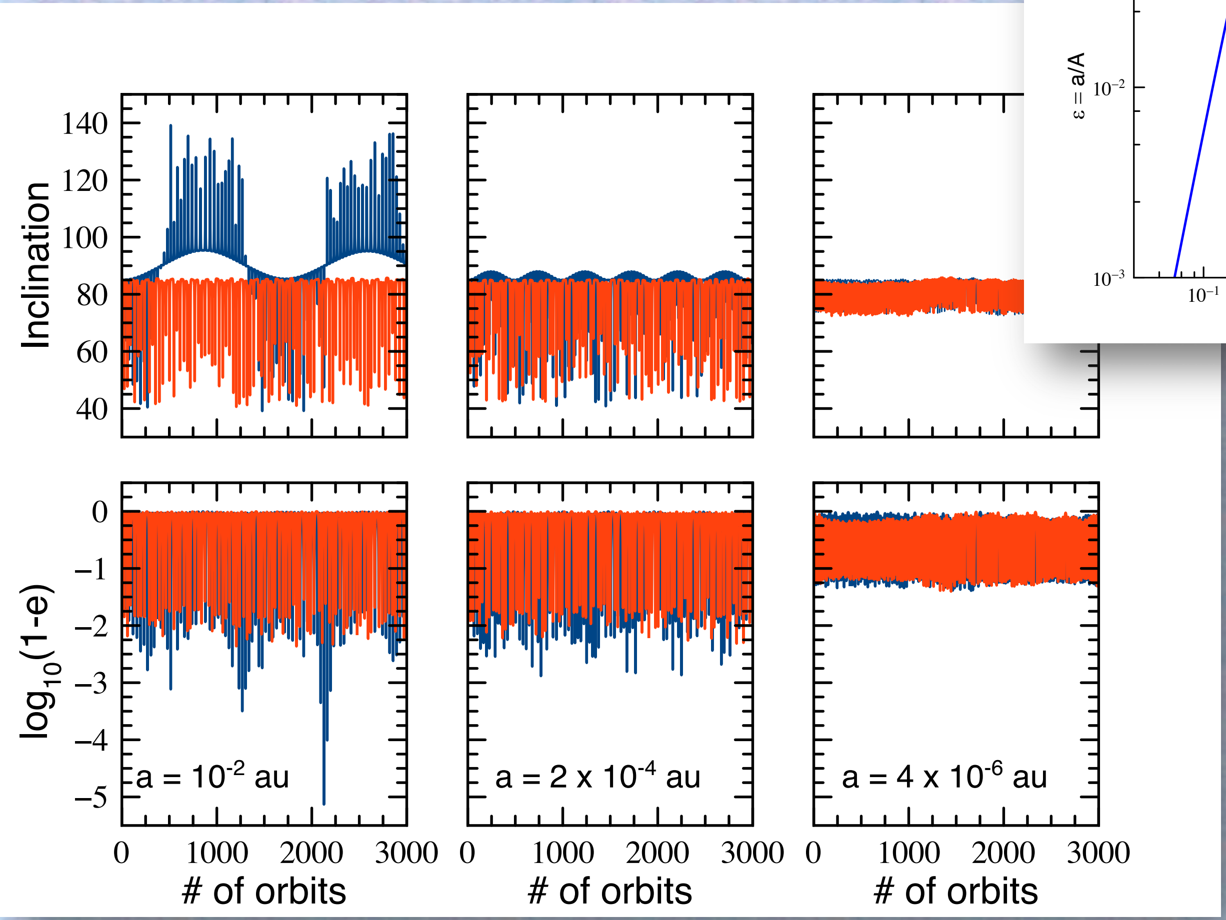
$$\frac{d\varpi}{d\tau} = \frac{3\pi}{64}\frac{\alpha^2\epsilon^{9/2}}{(1+\alpha)^{1/2}}\frac{1}{(1-E^2)^3}\Big[(3+2E^2)\cos z\Big(64-99e^2+3(12-17e^2)\cos^2 z+15(2-3e^2)\sin^2 z\cos 2\omega\Big)$$
$$-\frac{5}{2}E^2H(E)\Big(5(2-3e^2)[(1+\cos z)^2(2-3\cos z)\cos(2\omega-2\omega_3)-(1-\cos z)^2(2+3\cos z)\cos(2\omega+2\omega$$
$$-\,6(12-17e^2)\sin^2 z\cos z\cos 2\omega_3\Big)\Big],$$

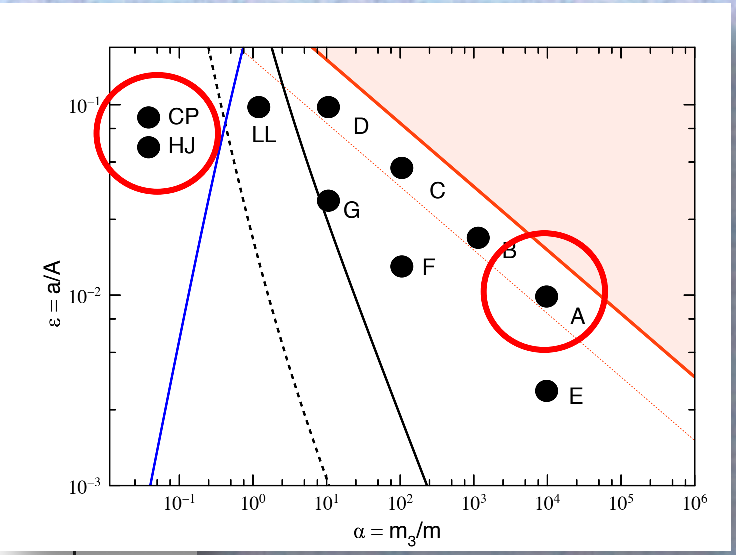
$$H(E) = 1-\frac{2(1-E^2)}{5(1+\sqrt{1-E^2})^2}$$

Agrees with Luo, Katz & Dong (2016)

Astrophysical implications of Q^2 terms

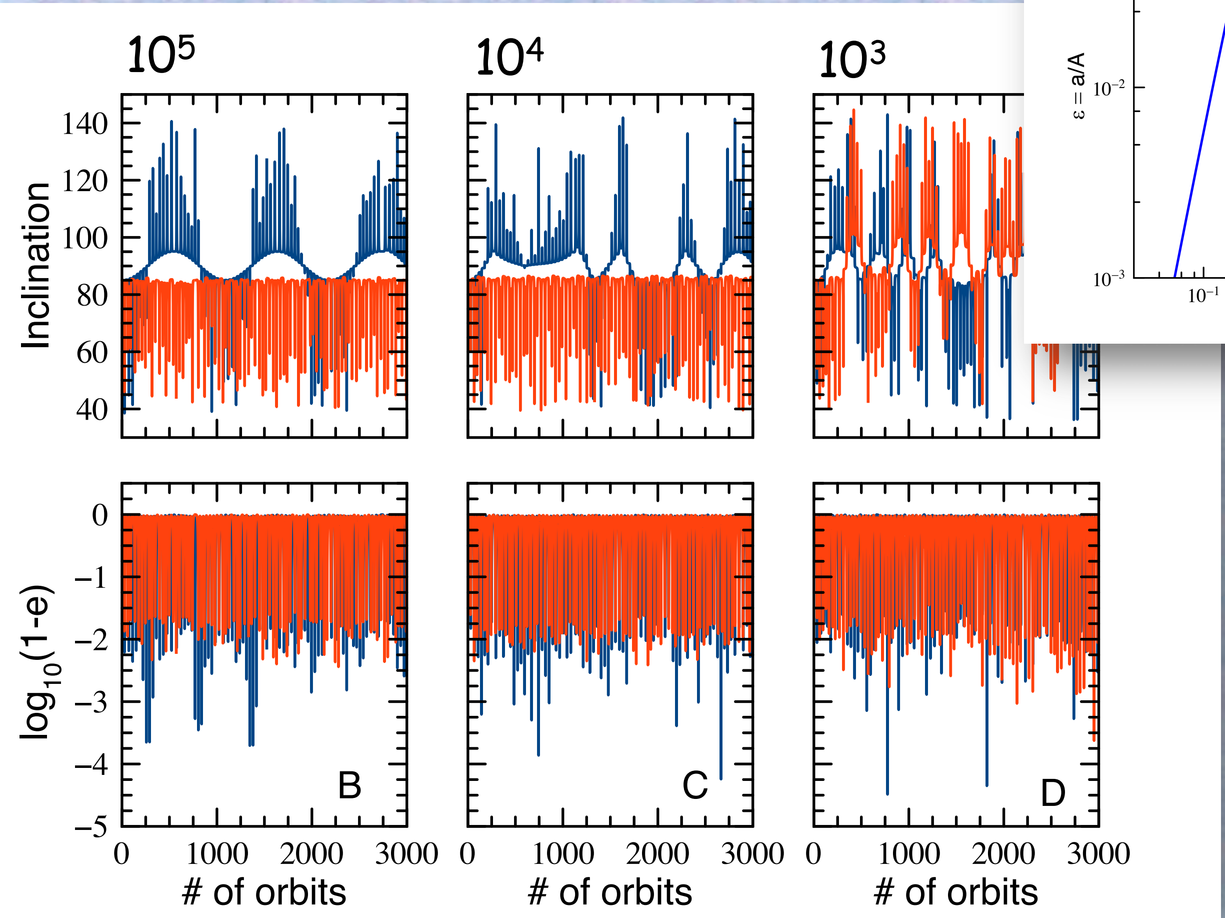
Case A: (1 + 100) + 10^6

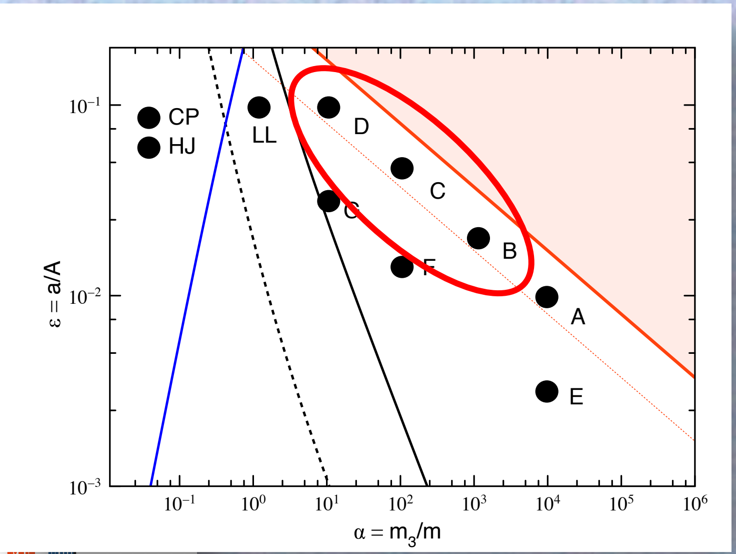




Astrophysical implications of Q^2 terms

Cases B, C & D: (1 + 100) + 10^N

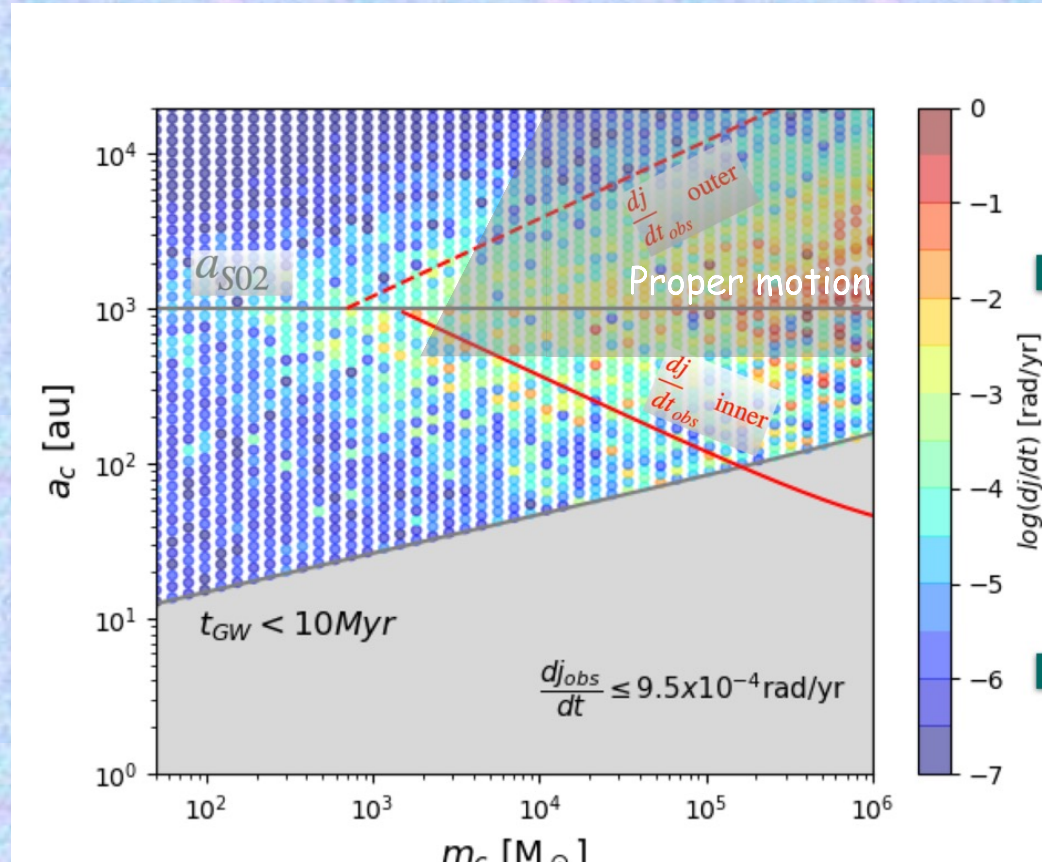




Application: A companion for SgrA*?

Naoz, CMW et al (2020)
Naoz, CMW & Zhang, in prep

- Companion IMBH would perturb the orbit of SO-2
- Orbital plane (i,Ω) & pericenter (ω) variations
- Use quadrupole-order Kozai-Lidov perturbations
- Observed bounds from UCLA Galactic Center group are ~ 10^{-3} rad/yr



Rotationally invariant quantities:

$$|dJ/dt|^2 = (di/dt)^2 + \sin^2 i (d\Omega/dt)^2 = \frac{3\pi\eta}{4P_\star}\left(\frac{a_c}{A_\star}\right)^2 \sin i_\star \frac{\mathcal{R}(e_c,\omega_c,i_\star)}{(1-e_\star^2)}$$

$$d\varpi/dt = (d\omega/dt) + \cos i (d\Omega/dt) = \frac{3\pi\eta}{8P_\star}\left(\frac{a_c}{A_\star}\right)^2 \frac{\mathcal{S}(e_c,\omega_c,i_\star)}{(1-e_\star^2)}$$

FLORIDA GRAVITY

Post-Newtonian cross-terms

Lagrange planetary equations:

$$\frac{dX_\alpha}{dt} = \epsilon Q_\alpha^{(3)}(X_\beta(t), t) + \delta Q_\alpha^{(PN)}(X_\beta(t), t)$$

$$X_\beta = \tilde{X}_\beta + \epsilon X_\beta^{(3)}(t) + \delta X_\beta^{(PN)}(t)$$

$$+ \epsilon\delta Q_\alpha^{(EIH)}(X_\beta(t), t)$$

$$\frac{df}{dt} = \frac{\sqrt{mp}}{r^2} - \epsilon\left(\frac{d\omega}{dt} + \cos i \frac{d\Omega}{dt}\right)^{(3)} - \delta\left(\frac{d\omega}{dt} + \cos i \frac{d\Omega}{dt}\right)^{(PN)}$$

Mercury's perihelion & PN cross terms

CMW, PRL **120**, 191101 (2018)

$$\Delta\omega = \frac{6\pi G m}{c^2 a(1-e^2)} + \sum_k \frac{3\pi}{2}\frac{m_k}{m}\left(\frac{a}{A_k}\right)^3 \frac{(1-e^2)^{1/2}}{(1-E_k^2)^{3/2}}\left[1 - \left(\frac{Gm}{c^2 a}\right)\frac{25 - 11\sqrt{1-e^2}}{(1-e^2)^{1/2}}\right]$$

PN, 43 as/c

Nbody, 532 as/c

PN cross term, 4×10^{-7}

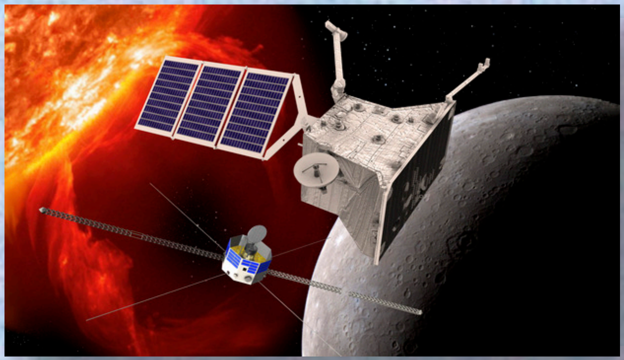
$$\Delta\omega_{\rm Current} \simeq 2 \times 10^{-3}\ {\rm as/c}$$

$$\Delta\omega_{\rm Cross} \simeq 2 \times 10^{-4}\ {\rm as/c}$$

$$\Delta\omega_{\rm BepiColombo} \simeq 4 \times 10^{-5}\ {\rm as/c}$$

$$\Delta\omega_{\rm 2PN} \simeq 3 \times 10^{-6}\ {\rm as/c}$$

With Bepi-Colombo, launched 20/10/18, data will be sensitive to PN cross terms



Adventures of a general relativist in an N-body world

- Brief history of the 3-body problem
- Hierarchical triple systems
- Higher multipoles and quadrupole-squared terms
- A companion black hole for SgrA*?
- Quadrupole-PN cross-terms and Mercury's perihelion
- Ongoing and future work
  - Complete "dotrio" terms (with Landen Conway)
  - Update bounds on SgrA* friend (with Smadar Naoz)
  - Test multipole expansion vs numerical solutions